Analytical and Numerical Approaches to Modeling Rogue Waves

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0.1 Introduction

Rogue, or ‘freak’, waves are powerful but rarely observed geophysical phenomena. These waves occur seemingly unannounced and have amplitudes much larger than the surrounding ocean water. Once considered legends and stories, these waves have historically not been well-understood. Only since last half of the twentieth century has the scientific community gathered official data and begun intense research on the subject.

These waves present a danger for passenger, industrial, and military sea-faring ships. Since the late 1960’s, there have been reports of more than 22 super-carriers being lost to rogue waves[5]. Furthermore, there are numerous recordings of these waves threatening offshore and coastal structures. Indeed, the main source of evidence regarding these extreme events come from offshore oil-platforms.

This paper presents several geophysical explanations for the formation of these waves, many of which are not well understood. Then, we present different mathematical approaches to the phenomenon including various analytical and numerical methods. Ultimately, with the lack of evidence and laboratory experiments, the mathematical and geophysical explanations for rogue waves is still infantile.

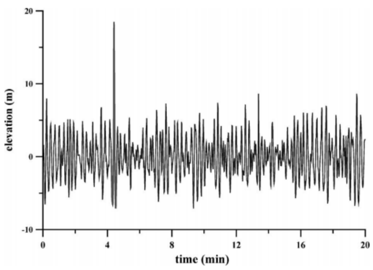

Figure 0.1: Rogue Wave in North Sea[7]

0.2 Physical Explanations

The main initial difficulty in formulating the problem is defining what is meant by ‘rogue waves’. There are several definitions that have different practical purposes. One definition means to compare the wave height - distance from the trough to crest - to the ’significant wave height’ - four times the standard deviation of the average surface elevation. If the ratio of wave height to significant wave height exceeds twice as tall, then the wave is considered rogue[5]. In other words, the rogue wave must be much taller than what would be statistically likely. This definition, however, is not totally accepted because it is highly dependent on the numerical context[8]. If the significant wave height happened to be a few centimeters, than a rogue wave need not be more than 20 centimeters in order to meet this definition.

Further complicating research is that rogue waves are very difficult to collect data on experimentally. The “Draupner wave” measured in the North Sea on January 1st, 1995, is the first experimentally successful documentation of rogue waves. Despite more recent data, there is still not enough to perform good analysis. Recorded events rarely last longer than a quarter of an hour, providing insufficient detail of the full statistical information of the sea[8]. Recently, satellite and radar measurements have provided researchers with photographic images of the ocean that are able to resolve individual waves. There are many concerns regarding the validity and reliability of such measurements but it remains of the major directions for future research in the area [7]

There are several hypotheses for explaining the mechanisms that can lead to rogue waves. Many of them revolve around the various linear and nonlinear focusing of the fluids but there is still uncertainty towards which factors are dominant. One of the major considerations is the process of Spatial Focusing - in which refracting waves align crests and effectively achieve constructive interference. The energy associated with such focusing is concentrated and leads to enormous rises in wave amplitude. Another linear mechanism that can lead to rogue waves is atmospheric forcing. Fluctuations in atmospheric pressure and shear wind flow can increase the wave energy significantly [7].

While the preceding theories are important in understanding basic geophysical interactions of the waves, nonlinear explanations are the main focus of current research. One of the main explanations for rogue waves comes from what is known as Benjamin-Feir instability - the unstable reaction of periodic wave trains to modulations [5]. As this instability in grows, wave trains dissolve and isolated waves can be produced. These fractured waves occasionally make their own groups and subsequent focusing creates large wave amplitudes. While these instabilities have been experimentally shown to create rogue waves, the extant conditions for instability is considered to be highly unlikely in natural circumstances [5]. BF instability appears to be related to the depth of the water. Instability theory appears to suggest that shallow water has a defocussing effect and prevents such unstable waves. Despite this theory, rogue waves in near-shore and shallow water environments have been experienced[8]. Further nonlinear properties of waves is studied in-depth numerically, which will be discussed later, but, ultimately, theories of rogue waves in general settings are not comprehensive. From a data perspective, rogue waves still appear to be random superposition and cannot be predicted from knowing the wave spectrum. “Because the underlying physical mechanisms for the generation of true rogue waves have not clearly been identified, there is not generally accepted prediction scheme”[5].

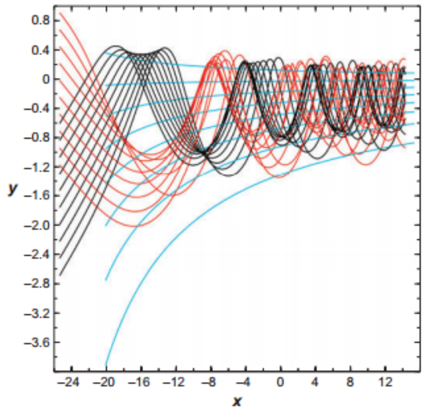

Figure 0.2: Spatial focusing and aligning of wave crests[5]

0.3 Mathematical Models

Because of their rarity in nature and difficulty in explaining their formation, scientists and researchers have looked to possible mathematical models exhibiting similar behavior. Hypothesized nonlinear properties of these waves lead interests to focus on nonlinear hyperbolic partial differential equations. The most studied equation is the Nonlinear Schrödinger Equation (NLSE). This equation is of particular interest because it admits soliton solutions that have the same properties found in rogue waves. In addition, analytical solutions, or ’breathers’, have been found in certain situations, leading to greater analytical tractability. This equation is not chosen, however, from any geophysical derivations or laws (like the Navier-Stokes equations). Nevertheless, it is the most studied equation in the field. Because of the inherent nonlinearity of those equations, numerical modeling is still the general route for analysis. This section will discuss the various analytical and numerical methods for studying these solutions and their applications to rogue waves.

0.3.1 Nonlinear Schrödinger Equation

Bearing similar resemblance and properties as the linear Schrödinger equation, the NLSE in one spatial dimension it takes the following form:

$$i\frac{\partial\psi}{\partial x} + \frac{1}{2}\frac{\partial^2\psi}{\partial t^2} + |\psi|^2\psi = 0$$

One reason that this particular equation has attracted a lot of attention is because it is known to admit soliton waves as solutions. Solitons, waves that traverse without losing energy when colliding with other waves, are suspected to closely resemble rogue waves. Further lending itself to easy analysis, the NLSE is known to admit solutions given arbitrary initial conditions and periodic boundary conditions[1].

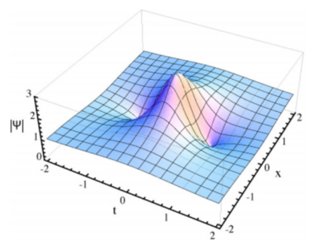

Figure 0.3: Spatial Rational Solution to NLSE[2]

Scientists suspect that rogue waves begin innocuously and must increase their amplitude exponentially until hitting a maximum and subsequently decay and disappear unto the sea. This general form, as shown on the left, is quite common among solutions. One particular class of waves admitted by the NLSE are called rational solutions and take the form

$$\psi = \left[1 - \frac{G + iH}{D}\right]e^{ix}$$

where G,H, and D are some combination of polynomials in the spatial and temporal variables. These and other rational solutions are well studied and present a tractable way to compute other properties of the wave such as intensity[2]. Much of the analytical research into the NLSE is surrounding the various ways of constructing higher-order solutions (higher order in terms of the polynomials G,H, and D) and more accurate representations. Recent progress has been made by [2] and [6] using Darboux Transformations - mappings between linear equation representations of the NLSE - to obtain better results.

Historically, most of the mathematical modeling of the NLSE has been done numerically. The nonlinear aspects of the equation, while challenging to confront numerically, make it relatively intractable analytically in the realistic circumstances. Numerical methods for finding solutions are divided into two rough groups: finite difference methods, and function approximations.

Finite difference methods start by constructing a rectangular grid in time and space and approximating the differential operators by algebraic relationships between grid points. Methods are divided between explicit - each additional time step is dependent upon previous grid points - and implicit methods - each time step is solved using an equation involving the current and later time steps. Implicit methods are particularly useful when the underlying system exhibits unstable behavior. For the schrödinger equations, implicit methods are commonly used to calculate the linear part and explicit methods for the nonlinear part. In addition, the Ablowitz and Ladik scheme is a popular choice for the NLSE [9]. More recent research has been implemented in order to create higher-order methods [4].

Function approximation methods estimates the exact solution by an approximate solution on a finite dimensional subspace. In other words, if $u(x,t)$ solves the NLSE then

$$u(x,t) \approx \sum_{k=1}^{n} \mu_k(t)\phi_k(x)$$

where the ϕ_k forms some sort of basis in the chosen subspace. Common transformations include various trigonometric approximations [3]. The most popular choice of methods is called the Split-Step Fourier Method. This method transforms the exact solution into Fourier space, advances in time based on the nonlinear part, solves the NSLE exactly, and transforms back using the fast Fourier transform (FFT) [9]. This method is popular because it is second order accurate in time and unconditionally stable. Another function approximation technique is based on the Pseudospectral Method - where the equation is transformed into a Fourier space and the derivatives are treated in an algebraic manner in the transformed variable. These methods are often faster and record lower error than their finite difference counterpart.

Because numerical solutions for rogue waves are usually studied in an isolated fashion, there is no known method for implementing this findings in a broader circulation model. There is also no efforts to combine numerical modeling of rogue waves with the NLSE with other common fluid dynamic equations such as the Navier-Stokes equations.

0.4 Conclusion

Rogue waves are of particular interest because of their potential devastating effects on ocean liners and structures. These phenomena are unpredictable and rare, making them notoriously difficult to study in the field. As a result, geophysical explanations for these waves are limited in scope and do not account for the formation of these waves in various settings. Researchers have looked to nonlinear wave equations in order to find potential candidates admitting solutions similar to those experienced in nature. The soliton solutions admitted by the Nonlinear Schrödinger equation have analytical representations - making them tractable - and match the behavior of real rogue waves. We still do not know to what extent we can apply the solutions of this equation to understanding rogue waves. Similar behavior found in financial markets [10] and nonlinear optics [1] make it increasingly attractive. Numerical solutions to the NLSE have focused primarily on functional approximation methods due to their speed and accuracy. Potential for future research include the need for additional linkage between the NLSE and rogue wave phenonmena, as well as investigation into the creation of rogue waves in shallow water.

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